Portfolio Theory

Suppose you are interested in holding a portfolio of two securities.

- Stock A has an expected return of 20% and standard deviation of 23%.
- Stock B has an expected return of 12% and standard deviation of 16%.

Suppose we want to allocate equal wealth in each security. What is the expected return of the portfolio (AB)? What is its standard deviation?

**Expected Return on a Portfolio of Stocks**

\[ \bar{R}_p = \sum_j w_j \bar{R}_j \]

where \( w_j \) = fraction of wealth invested in each security \( j \).

\[ R(AB) = 0.5 \times 0.20 + 0.5 \times 0.12 = 16\% \]
**Portfolio Variance:**

The general formula for the portfolio variance is given by:

$$\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} w_j w_i \sigma_j \sigma_i \rho_{ji}$$

where $\rho_{ji}$ is the correlation coefficient between returns $j$ and $i$.

For a portfolio with two stocks, this becomes:

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2w_1 w_2 \sigma_1 \sigma_2 \rho_{1,2}$$

The covariance measures how the two returns vary together, i.e., when stock A goes up, what stock B does. The correlation coefficient simply scales that measure over the overall variation of the securities.
Portfolio Diversification

The following graph tells the story. Suppose we randomly selected a stock and plotted its (in this case, monthly) standard deviation. Now we randomly draw another stock and plot the standard deviation of the equally weighted portfolio. We continue the exercise. Just by randomly selecting stocks we can decrease portfolio variance.

![Graph showing standard deviation of portfolio return as a function of number of stocks in portfolio.](image)

Note that the individual standard deviation is huge. Adding additional stocks quickly drives the portfolio standard deviation down.
Efficient Portfolios

1. Any investor who chooses to hold a portfolio a given variance will want the portfolio that has the maximum possible return among those portfolios that have the same variance.

2. Similarly, any investor who chooses a portfolio with a given mean return will want the portfolio with the minimum variance possible among those with the same mean return.

A portfolio that satisfies these conditions is known as an efficient portfolio. A portfolio is inefficient if there exists another portfolio with:

a. higher return for the same level or risk;
b. lower risk for the same return;
The Efficient Frontier

Previously, we found that as long as the securities are not perfectly correlated, a portfolio of the two stocks has lower variance than the average variance of the securities (diversification).
- **minimum variance frontier**: Among all portfolio combinations, the points farthest to the left have minimum variance.

- **efficient frontier**: The positively sloped portion. Portfolios on this frontier are referred to as mean-variance efficient (or efficient portfolios). These portfolios maximize the expected return on the portfolio for a given variance.

  No other portfolio with the same expected return has a lower standard deviation of return.

  No other portfolio with the same standard deviation of return has a higher expected return.
Suppose we add additional securities to the portfolio. This can only improve the minimum variance frontier.

Portfolio $Z$ is called the **minimum-variance portfolio**. Here you can find the minimum-variance frontiers between several different portfolios.
<table>
<thead>
<tr>
<th>Number of Stocks in Portfolio</th>
<th>Average Standard Deviation of Annual Portfolio Returns</th>
<th>Ratio of Portfolio Standard Deviation to Standard Deviation of a Single Stock</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>49.24%</td>
<td>1.00</td>
</tr>
<tr>
<td>10</td>
<td>23.93</td>
<td>0.49</td>
</tr>
<tr>
<td>50</td>
<td>20.20</td>
<td>0.41</td>
</tr>
<tr>
<td>100</td>
<td>19.69</td>
<td>0.40</td>
</tr>
<tr>
<td>300</td>
<td>19.34</td>
<td>0.39</td>
</tr>
<tr>
<td>500</td>
<td>19.27</td>
<td>0.39</td>
</tr>
<tr>
<td>1,000</td>
<td>19.21</td>
<td>0.39</td>
</tr>
</tbody>
</table>

These figures are from Table 1 in Meir Statman, “How Many Stocks Make a Diversified Portfolio?” *Journal of Financial and Quantitative Analysis* 22 (September 1987), pp. 353–64. They were derived from E. J. Elton and M. J. Gruber, “Risk Reduction and Portfolio Size: An Analytic Solution,” *Journal of Business* 50 (October 1977), pp. 415–37.
Average annual standard deviation (%)

Number of stocks in portfolio

Diversifiable risk

Nondiversifiable risk
CAPM

Total risk consists of *systematic risk* (which is priced or rewarded by investors) and *diversifiable risk* (which is not rewarded).

**Definition:**

Unsystematic (diversifiable) risk can be eliminated by diversification, so a portfolio with many assets has no (or almost no) systematic risk.
Risk in a Portfolio Sense

We now consider an investor who holds a widely diversified portfolio and is considering adding a new asset to her portfolio. How will this new asset affect the risk of this portfolio?

- A new asset adds to the risk of the portfolio through its covariance with the portfolio, \( \text{cov}(R_i, R_m) \)

- A new asset adds to the return of the portfolio through its own return.

The incremental expected return and risk of a single security are related as:

\[
E[R_j] = r_f + \beta_j [E(R_m) - r_f]
\]

where

\[
\beta_j = \frac{\text{cov}(r_j, r_m)}{\sigma_m^2}
\]
Beta measures the sensitivity of a stock to changes in the value of the large portfolio. It measures how much systematic risk a particular asset has relative to the average asset.

Note that the beta coefficient of the market portfolio itself is:

\[
\beta_m = \frac{\text{cov}(r_m, r_m)}{\sigma_m^2} = \frac{\sigma_m^2}{\sigma_m^2} = 1
\]
An Example

Suppose you are a banker considering an equity investment in a company with a new drug process. The process is inherently risky, i.e. the standard deviation of the project is 75% per year. The beta of the project is 0.5. The Rf = 5% and the E[Rm] = 13.5%. What is the required rate of return on the project?

CAPM tells us that the answer depends on the measure of the systematic risk -- the beta of the project.

\[ E[R_{drug}] = 5\% + (.5)(13.5\% - 5\%) = 9.25\% \]

This is the required rate of return on the project. The \( \beta \) is the only relevant piece of information -- now all that remains is to estimate it!
How Do You Estimate $\beta$?

We want to know how the return of a security changes in response to future changes in the market portfolio. This is more difficult than it seems, since we cannot readily identify the market portfolio. In practice analysts typically use the return on S&P 500 as a proxy for the return on the market portfolio. To estimate beta, regress the security returns for the past on the market returns.

$$R_{i,t} = \alpha_i + \beta_i R_{m,t} + e_{i,t}$$

The slope in this regression is an estimate of $\beta$. 

![Diagram showing the relationship between security return and S&P return, with beta as the slope of the line.]
Dell Computer

Price data – Feb 95 – Jul 01

\[ R^2 = 0.27 \]
\[ \beta = 2.02 \]

Slope determined from plotting the line of best fit.

Market return (%)
General Motors

Price data – Feb 95 – Jul 01

$R^2 = .25$

$\beta = 1.00$

Slope determined from plotting the line of best fit.

Market return (%)
Exxon Mobil

Price data – Feb 95 – Jul 01

\[ R^2 = 0.16 \]

\[ \beta = 0.42 \]

Slope determined from plotting the line of best fit.
Once we have estimated beta, we can estimate the expected (required) return on the security:

We get the Market Risk Premium (MRP), from historical data:

It has been \( 8.5\% \) relative to T-bills
and \( 7.5\% \) relative to T-bonds

Finally, find the \( r_f \) in the newspaper

Plug the MRP, \( r_f \) and beta in the CAPM equation.
These are the betas for some companies:

<table>
<thead>
<tr>
<th>Company</th>
<th>Beta Coefficient ($\beta_i$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exxon</td>
<td>0.65</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>0.90</td>
</tr>
<tr>
<td>IBM</td>
<td>0.95</td>
</tr>
<tr>
<td>Wal-Mart</td>
<td>1.10</td>
</tr>
<tr>
<td>General Motors</td>
<td>1.15</td>
</tr>
<tr>
<td>Microsoft</td>
<td>1.30</td>
</tr>
<tr>
<td>Harley-Davidson</td>
<td>1.65</td>
</tr>
<tr>
<td>America Online</td>
<td>2.40</td>
</tr>
</tbody>
</table>
Portfolio Betas

One remarkable fact that comes from the linearity of this equation is that we can obtain the beta of a portfolio of assets by simply multiplying the betas of the assets by their portfolio weights.

Example:
The beta of a 50/50 portfolio of two assets, one with a beta of .8 and the other with a beta of 1 is: 0.5 x 0.8 + 0.5 x 1 = 0.9.

<table>
<thead>
<tr>
<th>Stock</th>
<th>Amount Invested</th>
<th>Portfolio Weights</th>
<th>Beta</th>
<th>(3) x (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T</td>
<td>$6,000</td>
<td>50%</td>
<td>0.90</td>
<td>0.450</td>
</tr>
<tr>
<td>Wal-Mart</td>
<td>4,000</td>
<td>33%</td>
<td>1.10</td>
<td>0.367</td>
</tr>
<tr>
<td>Microsoft</td>
<td>2,000</td>
<td>17%</td>
<td>1.30</td>
<td>0.217</td>
</tr>
<tr>
<td>Portfolio</td>
<td>$12,000</td>
<td>100%</td>
<td></td>
<td>1.034</td>
</tr>
</tbody>
</table>
Capital Market Line

Return

Market Return = \( r_m \)

Risk Free Return = \( r_f \)

Efficient Portfolio

Risk (\( \sigma \))
Security Market Line

Market Return = $r_m$

Risk Free Return = $r_f$

Return

Efficient Portfolio

Risk ($\beta$)

1.0
Security Market Line

\[ \text{Risk Free Return} = r_f \]

Return

Risk (\(\beta\))

Security Market Line (SML)
SML Equation $= r_f + \beta (r_m - r_f)$