Portfolio Theory

Suppose you are interested in holding a portfolio of two securities.

- Stock A has an expected return of 20% and standard deviation of 23%.
- Stock B has an expected return of 12% and standard deviation of 16%.

Suppose we want to allocate equal wealth in each security. What is the expected return of the portfolio (AB)? What is its standard deviation?

Expected Return on a Portfolio of Stocks

\[
\bar{R}_p = \sum_j w_j \bar{R}_j
\]

where \( w_j \) = fraction of wealth invested in each security \( j \).

\[R(AB) = 0.5 \times 0.20 + 0.5 \times 0.12 = 16\%\]
**Portfolio Variance:**

The general formula for the portfolio variance is given by:

\[
\sigma_p^2 = \sum_{i=1}^{N} \sum_{j=1}^{N} \omega_i \omega_j \sigma_i \sigma_j \rho_{ji}
\]

where \( \rho_{ji} \) is the correlation coefficient between returns \( j \) and \( i \).

The covariance measures how the two returns vary together, i.e., when stock A goes up, what stock B does. The correlation coefficient simply scales that measure over the overall variation of the securities.
Covariance of the return:

Suppose you are given two assets, A and B. The covariance measures how the returns of the two assets vary together in the different states of the world (what happens to A as B move up or down, relative to their means)

If we deal with historical data:

$$\text{cov}(R_a, R_b) = \frac{1}{n-1} \sum_i (R_{i,a} - \bar{R}_a)(R_{i,b} - \bar{R}_b)$$

If we can assess the actual probabilities of each state of the world

$$\text{cov}(R_a, R_b) = \sum_i [R_{i,a} - E(R_a)][R_{i,b} - E(R_b)]\Pr ob_i$$

Note that E[Ra] and E[Rb] are the expected returns, which we have specified previously.
Correlation coefficient:

This is simply the covariance between the two securities, scaled by their standard deviations:

\[ \rho_{ji} = \frac{\text{cov}(R_i, R_j)}{\sigma_i \sigma_j} \]

For our two securities, the portfolio variance becomes:

\[ \sigma^2 = (w_1^2 \sigma_1^2 + w_1w_2 \sigma_1 \sigma_2 \rho_{1,2} + w_2w_1 \sigma_2 \sigma_1 \rho_{2,1} + w_2^2 \sigma_2^2), \text{ or} \]

\[ \sigma^2 = (w_1^2 \sigma_1^2 + w_1w_2 \text{cov}(R_1, R_2) + w_1w_2 \text{cov}(R_1, R_2) + w_2^2 \sigma_2^2) \]

Again, the standard deviation, \( \sigma \), is simply the square root of the variance.

The correlation coefficient is an important variable. Let's vary the weights in stock A and stock B, and calculate the expected return and the standard deviation.

Let’s first look at the case when \( \rho_{1,2} = 1 \). We want to plot the portfolio return against its standard deviation for various weights.
Note that when $\rho_{1,2} = 1$, the expected return remains the same, but the portfolio standard deviation becomes:

$$\sigma_p = w_1 \sigma_1 + w_2 \sigma_2,$$

or simply a linear combination of the standard deviations of the two securities;
<table>
<thead>
<tr>
<th>Weight(A)</th>
<th>Weight(B)</th>
<th>R(AB)</th>
<th>St.Dev(AB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.12</td>
<td>0.16</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>0.128</td>
<td>0.167</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.136</td>
<td>0.174</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7</td>
<td>0.144</td>
<td>0.181</td>
</tr>
<tr>
<td>0.4</td>
<td>0.6</td>
<td>0.152</td>
<td>0.188</td>
</tr>
<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.16</td>
<td>0.195</td>
</tr>
<tr>
<td>0.6</td>
<td>0.4</td>
<td>0.168</td>
<td>0.202</td>
</tr>
<tr>
<td>0.7</td>
<td>0.3</td>
<td>0.176</td>
<td>0.209</td>
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<tr>
<td>0.8</td>
<td>0.2</td>
<td>0.184</td>
<td>0.216</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1</td>
<td>0.192</td>
<td>0.223</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.2</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Where is the plot when the whole portfolio consists of Stock B only?
Where is the plot when the whole portfolio consists of Stock A only?
Where is the plot when the whole portfolio consists of 50% Stock A and 50% stock B?
Where is the plot when the whole portfolio consists of 20% Stock A and 80% stock B?
Now, let’s look at a more realistic case, when \(-1 < \rho_{A,B} < 1\). Let \(\rho_{A,B} = 0.4\)

Where is the plot when the whole portfolio consists of Stock B only?
Where is the plot when the whole portfolio consists of Stock A only?
Where is the plot when the whole portfolio consists of 50% Stock A and 50% stock B?
Where is the plot when the whole portfolio consists of 20% Stock A and 80% stock B?
<table>
<thead>
<tr>
<th>Weight(A)</th>
<th>Weight(B)</th>
<th>R(AB)</th>
<th>St.Dev(AB)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>0.12</td>
<td>0.16</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>0.128</td>
<td>0.154643</td>
</tr>
<tr>
<td>0.2</td>
<td>0.8</td>
<td>0.136</td>
<td>0.15235</td>
</tr>
<tr>
<td>0.3</td>
<td>0.7</td>
<td>0.144</td>
<td>0.153256</td>
</tr>
<tr>
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<td>0.6</td>
<td>0.152</td>
<td>0.157307</td>
</tr>
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<td>0.5</td>
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<td>0.164271</td>
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<td>0.3</td>
<td>0.176</td>
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<td>0.192</td>
<td>0.213903</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0.2</td>
<td>0.23</td>
</tr>
</tbody>
</table>

What happens to \( \sigma \) and \( R \) as we change the weights? Let’s look at the case where weight (A) = 20% and weight (B)=80%. We added to stock B (80%) a security with higher risk, stock A (20%), and yet, the overall portfolio risk was reduced!
This result demonstrates that the portfolio variance is lower than the variance of the individual assets. So combining stocks that have less than perfect positive correlation is a strategy that will reduce the variance of the returns on your portfolio. This is called diversification.

In the extreme case, when \( \rho_{A,B} = -1 \), the portfolio standard deviation becomes:

\[
\sigma_p = |w_1 \sigma_1 - w_2 \sigma_2|
\]

and therefore can be driven down to zero, for certain weights, \( w_1 \) and \( w_2 \).

Solving for the weights at which \( \sigma_p = 0 \):

\[
|w_1 \sigma_1 - w_2 \sigma_2| = 0 \Rightarrow w_1 \sigma_1 = w_2 \sigma_2
\]

and \( w_1 + w_2 = 1 \) \( \Rightarrow w_2 = 1 - w_1 \)

Substituting for \( w_2 \),

\[
w_1 \sigma_1 = (1 - w_1) \sigma_2 \\
w_1 \sigma_1 = \sigma_2 - w_1 \sigma_2 \\
w_1 (\sigma_1 + \sigma_2) = \sigma_2 \\
w_1 = \sigma_2 / (\sigma_1 + \sigma_2) , \text{ and therefore, } w_2 = 1 - \sigma_2 / (\sigma_1 + \sigma_2) = \sigma_1 / (\sigma_1 + \sigma_2)
\]
The following graph tells the story. Suppose we randomly selected a stock and plotted its (in this case, monthly) standard deviation. Now we randomly draw another stock and plot the standard deviation of the equally weighted portfolio. We continue the exercise. Just by randomly selecting stocks we can decrease portfolio variance.

Note that the individual standard deviation is huge. Adding additional stocks quickly drives the portfolio standard deviation down.
Efficient Portfolios

1. Any investor who chooses to hold a portfolio a given variance will want the portfolio that has the maximum possible return among those portfolios that have the same variance.

2. Similarly, any investor who chooses a portfolio with a given mean return will want the portfolio with the minimum variance possible among those with the same mean return.

A portfolio that satisfies these conditions is known as an efficient portfolio. A portfolio is inefficient if there exists another portfolio with:

   a. higher return for the same level or risk;
   b. lower risk for the same return;
The Efficient Frontier

Previously, we found that as long as the securities are not perfectly correlated, a portfolio of the two stocks has lower variance than the average variance of the securities (diversification).
- **minimum variance frontier**: Among all portfolio combinations, the points farthest to the left have minimum variance.

- **efficient frontier**: The positively sloped portion. Portfolios on this frontier are referred to as **mean-variance efficient (or efficient portfolios)**. These portfolios maximize the expected return on the portfolio for a given variance.

  No other portfolio with the same expected return has a lower standard deviation of return.

  No other portfolio with the same standard deviation of return has a higher expected return.
Suppose we add additional securities to the portfolio. This can only improve the minimum variance frontier.

Portfolio Z is called the **minimum-variance portfolio**. Here you can find the minimum-variance frontiers between several different portfolios.
Mean-Variance Frontier with Risk-free Asset

Now consider the introduction of a riskless security like a Treasury bill. Suppose we invest in a combination of a portfolio on the efficient set (derived without the riskless security) and the riskless security. We can calculate the expected return and the standard deviation on this new portfolio.

\[
\text{E}(R_{p_{rf}}) = (1-w) R_{rf} + w \text{E}(R_{p})
\]

\[
\sigma_{p_{rf}} = w \sigma_{p}
\]

Where \( rf \) represents the risk free security, \( p \) represents the portfolio on the efficient frontier and \( w \) is the proportion of funds invested in the risky security.
Which portfolio among all efficient portfolios are you going to choose?

It's the portfolio that maximizes the risk premium per unit of risk (measured by standard deviation):

\[
\frac{(E[R_p] - R_f)}{\sigma_p}.
\]

Effectively, we are drawing a tangent from the riskless point to the efficient frontier above.
Two-fund Separation

If all investors agree upon $E[R]$, $\sigma$ and $R_f$, then they all hold a combination of the stocks in portfolio M, and the risk-free security.

Since all stocks must be held by someone, all stocks would be in this portfolio. If all risky securities are in this portfolio, it must be the *market portfolio*.

The implication: Every investor holds two “funds” -- a fund comprised of the risk-free asset, and a fund which is the market portfolio. We call this the **two-fund separation theorem**.
CAPM

Total risk consists of *systematic risk* (which is priced or rewarded by investors) and *diversifiable risk* (which is not rewarded).

Definition:

Unsystematic (diversifiable) risk can be eliminated by diversification, so a portfolio with many assets has no (or almost no) systematic risk.

**Problem**

Which of the following risks of a stock are likely to be firm-specific, diversifiable risks, and which are likely to be systematic risks? Which risks will affect the risk premium that investors will demand?

a. The risk that the founder and CEO retires  
b. The risk that oil prices rise, increasing production costs  
c. The risk that a product design is faulty and the product must be recalled  
d. The risk that the economy slows, reducing demand for the firm’s products
What is the optimal portfolio M? It's the portfolio that maximizes the risk premium per unit of risk (measured by standard deviation):

\[
\frac{(E[R_p] - R_f)}{\sigma_p}.
\]
The Capital Market Line

The equation for the capital market line is:

\[ y = \text{intercept} + \text{slope of line} \times x \]

Here the intercept is the return to the riskless asset \( R_f \).

The slope of the line for any two coordinates \( (x_1, y_1) \), \( (x_2, y_2) \) on the line is given by:

\[ \frac{y_2 - y_1}{x_2 - x_1} \quad \text{(Rise over run).} \]

Here the two points on the line we can use are \( M \) and \( R_f \).

Therefore the equation of the capital market line is given by:

\[
E[R_p] = R_f + \frac{E(R_m) - R_f}{\sigma_m} \sigma_p
\]

or,

Expected Return = Risk-free Rate + (Risk Premium per unit of risk) x (Units of Risk)
Now, given that investors hold diversified portfolios, how do we measure the risk and return of an investor’s portfolio? Suppose that an investor has formed a portfolio of $n$ assets. That is, he has invested $1/n$ proportion of his wealth in each of $n$ different assets. We know that the expected return of such a portfolio is given by:

$$\bar{R}_p = \sum_j w_j \bar{R}_j = \frac{1}{n} (r_1 + r_2 + ... + r_n)$$

And the variance is:

$$\sigma^2_p = \sum_{i=1}^{N} \sum_{j=1}^{N} w_i w_j \sigma_i \sigma_j \rho_{ij} = \frac{1}{n^2} \sum_i \sigma_i^2 + \frac{1}{n^2} \sum_i \sum_j \text{cov}(R_i, R_j)$$

Note that there are:

- $n$ variance terms
- $n^2 - n$ covariance terms

so the variance equation simplifies to:
That is, the average covariance

\[ \sigma_p^2 = \frac{1}{n}[\text{Average Variance}] + (1 - \frac{1}{n})[\text{Average Covariance}] \]

When \( n \) become large, the portfolio variance becomes the \textit{average covariance}: That is, the variance of a large portfolio is simply the average covariance of the individual stocks. The variance of the individual stock adds little (or nothing) to the portfolio variance, but the stock’s covariance with the other securities becomes important. The risk that remains (and that is composed of the sum of covariances) is called the \textbf{systematic risk} and it cannot be eliminated by diversification.

**Definition:**

The expected return on an asset depends only on that asset’s systematic risk.
<table>
<thead>
<tr>
<th>Number of Stocks in Portfolio</th>
<th>Average Standard Deviation of Annual Portfolio Returns</th>
<th>Ratio of Portfolio Standard Deviation to Standard Deviation of a Single Stock</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>49.24%</td>
<td>1.00</td>
</tr>
<tr>
<td>10</td>
<td>23.93</td>
<td>0.49</td>
</tr>
<tr>
<td>50</td>
<td>20.20</td>
<td>0.41</td>
</tr>
<tr>
<td>100</td>
<td>19.69</td>
<td>0.40</td>
</tr>
<tr>
<td>300</td>
<td>19.34</td>
<td>0.39</td>
</tr>
<tr>
<td>500</td>
<td>19.27</td>
<td>0.39</td>
</tr>
<tr>
<td>1,000</td>
<td>19.21</td>
<td>0.39</td>
</tr>
</tbody>
</table>

These figures are from Table 1 in Meir Statman, “How Many Stocks Make a Diversified Portfolio?” Journal of Financial and Quantitative Analysis 22 (September 1987), pp. 353–64. They were derived from E. J. Elton and M. J. Gruber, “Risk Reduction and Portfolio Size: An Analytic Solution,” Journal of Business 50 (October 1977), pp. 415–37.
Average annual standard deviation (%)

Number of stocks in portfolio

Diversifiable risk

Nondiversifiable risk
Risk in a Portfolio Sense

We now consider an investor who holds a widely diversified portfolio and is considering adding a new asset to her portfolio. How will this new asset affect the risk of this portfolio?

- A new asset adds to the risk of the portfolio through its covariance with the portfolio, \( \text{cov}(R_i, R_m) \)

- A new asset adds to the return of the portfolio through its own return.

The incremental expected return and risk of a single security are related as:

\[
E[R_j] = r_f + \beta_j \left[ E(R_m) - r_f \right]
\]

where

\[
\beta_j = \frac{\text{cov}(r_j, r_m)}{\sigma_m^2}
\]
Beta measures the sensitivity of a stock to changes in the value of the large portfolio. It measures how much systematic risk a particular asset has relative to the average asset.

Note that the beta coefficient of the market portfolio itself is:

\[
\beta_m = \frac{\text{cov}(r_m, r_m)}{\sigma_m^2} = \frac{\sigma_m^2}{\sigma_m^2} = 1
\]
An Example

Suppose you are a banker considering an equity investment in a company with a new drug process. The process is inherently risky, i.e. the standard deviation of the project is 75% per year. The beta of the project is 0.5. The Rf = 5% and the E[Rm] = 13.5%. What is the required rate of return on the project?

CAPM tells us that the answer depends on the measure of the systematic risk -- the beta of the project.

\[ E[R_{drug}] = 5\% + (.5)(13.5\% - 5\%) = 9.25\% \]

This is the required rate of return on the project. The β is the only relevant piece of information -- now all that remains is to estimate it!
How Do You Estimate $\beta$?

We want to know how the return of a security changes in response to future changes in the market portfolio. This is more difficult than it seems, since we cannot readily identify the market portfolio. In practice analysts typically use the return on S&P 500 as a proxy for the return on the market portfolio. To estimate beta, regress the security returns for the past on the market returns.

$$R_{i,t} = \alpha_i + \beta_i R_{m,t} + e_{i,t}$$

The slope in this regression is an estimate of $\beta$. 

Once we have estimated beta, we can estimate the expected (required) return on the security:

How do we estimate the Market Risk Premium (MRP)?

From historical data:

It has been 8.5% relative to T-bills
and 7.5% relative to T-bonds

However, recent research shows that this is too high. We should use 4-6%.

Finally, find the $r_f$ in the newspaper

Plug the MRP, $r_f$ and beta in the CAPM equation.
These are the betas for some companies:

<table>
<thead>
<tr>
<th>Company</th>
<th>Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exxon</td>
<td>0.65</td>
</tr>
<tr>
<td>AT&amp;T</td>
<td>0.90</td>
</tr>
<tr>
<td>IBM</td>
<td>0.95</td>
</tr>
<tr>
<td>Wal-Mart</td>
<td>1.10</td>
</tr>
<tr>
<td>General Motors</td>
<td>1.15</td>
</tr>
<tr>
<td>Microsoft</td>
<td>1.30</td>
</tr>
<tr>
<td>Harley-Davidson</td>
<td>1.65</td>
</tr>
<tr>
<td>America Online</td>
<td>2.40</td>
</tr>
</tbody>
</table>
## Betas With Respect to the S&P 500 for Individual Stocks and Average Betas for Stocks in Their Industries (based on monthly data for 2000-2005)

<table>
<thead>
<tr>
<th>Industry</th>
<th>Beta</th>
<th>Ticker</th>
<th>Firm</th>
<th>Beta</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gold and Silver</td>
<td>-0.04</td>
<td>NEM</td>
<td>Newmont Mining Corporation</td>
<td>0.02</td>
</tr>
<tr>
<td>Beverages (Alcoholic)</td>
<td>0.23</td>
<td>BUD</td>
<td>Anheuser-Busch Companies, Inc.</td>
<td>0.10</td>
</tr>
<tr>
<td>Personal and Household Prods.</td>
<td>0.25</td>
<td>PG</td>
<td>The Procter &amp; Gamble Company</td>
<td>0.19</td>
</tr>
<tr>
<td>Food Processing</td>
<td>0.34</td>
<td>HNZ</td>
<td>H. J. Heinz Company</td>
<td>0.37</td>
</tr>
<tr>
<td></td>
<td></td>
<td>HSY</td>
<td>The Hershey Company</td>
<td>-0.10</td>
</tr>
<tr>
<td>Beverages (Nonalcoholic)</td>
<td>0.43</td>
<td>KO</td>
<td>The Coca-Cola Company</td>
<td>0.50</td>
</tr>
<tr>
<td>Electric Utilities</td>
<td>0.48</td>
<td>EIX</td>
<td>Edison International</td>
<td>0.50</td>
</tr>
<tr>
<td>Major Drugs</td>
<td>0.48</td>
<td>PFE</td>
<td>Pfizer Inc.</td>
<td>0.54</td>
</tr>
<tr>
<td>Restaurants</td>
<td>0.69</td>
<td>SBUX</td>
<td>Starbucks Corporation</td>
<td>0.60</td>
</tr>
<tr>
<td>Retail (Grocery)</td>
<td>0.74</td>
<td>SWY</td>
<td>Safeway Inc.</td>
<td>0.67</td>
</tr>
<tr>
<td>Conglomerates</td>
<td>0.84</td>
<td>GE</td>
<td>General Electric Company</td>
<td>0.85</td>
</tr>
<tr>
<td>Forestry and Wood Products</td>
<td>0.95</td>
<td>WY</td>
<td>Weyerhaeuser Company</td>
<td>0.96</td>
</tr>
<tr>
<td>Recreational Products</td>
<td>1.00</td>
<td>HDI</td>
<td>Harley-Davidson, Inc.</td>
<td>1.14</td>
</tr>
<tr>
<td>Apparel/Accessories</td>
<td>1.12</td>
<td>LIZ</td>
<td>Liz Claiborne, Inc.</td>
<td>0.90</td>
</tr>
<tr>
<td>Retail (Home Improvement)</td>
<td>1.22</td>
<td>HD</td>
<td>Home Depot, Inc.</td>
<td>1.43</td>
</tr>
<tr>
<td>Auto and Truck Manufacturers</td>
<td>1.44</td>
<td>GM</td>
<td>General Motors Corporation</td>
<td>1.20</td>
</tr>
<tr>
<td>Computer Hardware</td>
<td>1.60</td>
<td>AAPL</td>
<td>Apple Computer, Inc.</td>
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<tr>
<td>Software and Programming</td>
<td>1.74</td>
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<td></td>
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<td>AMD</td>
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<tr>
<td></td>
<td></td>
<td>INTC</td>
<td>Intel Corporation</td>
<td>2.17</td>
</tr>
</tbody>
</table>
**Problem**

Suppose that in the coming year, you expect Microsoft stock to have a volatility of 23% and a beta of 1.28, and McDonald’s stock to have a volatility of 37% and a beta of 0.99. Which stock carries more total risk? Which has more systematic risk? If the risk-free interest rate is 4% and the market’s expected return is 10%, estimate the cost of capital for a project with the same beta as McDonald’s stock and a project with the same beta as Microsoft stock. Which project has a higher cost of capital?
Portfolio Betas

One remarkable fact that comes from the linearity of this equation is that we can obtain the beta of a portfolio of assets by simply multiplying the betas of the assets by their portfolio weights.

Example:
The beta of a 50/50 portfolio of two assets, one with a beta of .8 and the other with a beta of 1 is: 0.5 x 0.8 + 0.5 x 1 = 0.9.

<table>
<thead>
<tr>
<th>Stock</th>
<th>Amount Invested</th>
<th>Portfolio Weights</th>
<th>Beta</th>
<th>(3) x (4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>AT&amp;T</td>
<td>$6,000</td>
<td>50%</td>
<td>0.90</td>
<td>0.450</td>
</tr>
<tr>
<td>Wal-Mart</td>
<td>4,000</td>
<td>33%</td>
<td>1.10</td>
<td>0.367</td>
</tr>
<tr>
<td>Microsoft</td>
<td>2,000</td>
<td>17%</td>
<td>1.30</td>
<td>0.217</td>
</tr>
<tr>
<td>Portfolio</td>
<td>$12,000</td>
<td>100%</td>
<td>1.03</td>
<td>1.034</td>
</tr>
</tbody>
</table>